

Orbital Functional Geometry XV

Orbital Space on Riemannian Manifolds:
Curvature, Fisher Information, and the Ricci Flow

Preprint

Judicael Brindel

Independent Researcher, Cotonou, Benin

ORCID: 0009-0007-4590-9874

March 2026

Abstract

We extend the Orbital Space $\mathcal{O}$ to functions $\psi : (M, g) \rightarrow \mathbb{R}^+$ on a Riemannian manifold (M, g) .

The orbital Laplacian is $\Delta_{\mathcal{O},g}\psi = \Delta_g(\ln \psi) = \Delta_g u$ where $u = \ln \psi$. The linearisation theorem holds on every Riemannian manifold: the substitution $u = \ln \psi$ transforms the orbital Schrödinger equation into the free equation $i\hbar\partial_t u = -(\hbar^2/2m)\Delta_g u$. The stationary states are $\psi_k = e^{v_k}$ where $\Delta_g v_k = -\lambda_k v_k$ are the eigenfunctions of the Laplace–Beltrami operator.

The orbital potential generalises to:

$$V_{\mathcal{O},g}(\psi, x) = \frac{\hbar^2}{2m} |\nabla_g \ln \psi|_g^2$$

By the Bochner identity, the Ricci curvature controls its evolution: if $\text{Ric} \geq K > 0$, the orbital potential grows faster under Δ_g (positive curvature localises orbital states); if $\text{Ric} \leq -K < 0$, it spreads (negative curvature delocalises).

The relation $\langle E \rangle = (\hbar^2/8m)I_g(p)$ between orbital energy and Fisher information holds on every Riemannian manifold. On a compact manifold of diameter d :

$$\langle E \rangle_g \geq \frac{\hbar^2 C(M, g)}{8m d^2}$$

giving geometric confinement: the smaller the manifold, the higher the minimum orbital energy.

Under the Ricci flow $\partial_t g = -2\text{Ric}(g)$, the mean orbital potential decreases on manifolds with positive Ricci curvature. On Einstein metrics $\text{Ric} = \lambda g$, the orbital potential is an eigenfunction of Δ_g .

Contents

1	Orbital Laplacian on (M, g)	4
2	Orbital Potential and Riemannian Geometry	4
3	Fisher Information on (M, g)	5
4	Orbital Space under the Ricci Flow	5
5	The Complete Hierarchy of $\mathcal{O}$	6

Introduction

Papers XIII–XIV applied the orbital framework to Fisher–KPP and extended $\mathcal{O}$ to $\mathbb{R}^n$. The natural next step is to allow the underlying space to be curved.

On a Riemannian manifold (M, g) , the flat Laplacian Δ is replaced by the Laplace–Beltrami operator Δ_g . The orbital construction carries through without modification: $u = \ln \psi$ still linearises the orbital Schrödinger equation, and the orbital potential $V_{\mathcal{O},g} = (\hbar^2/2m)|\nabla_g u|_g^2$ is still the local Fisher information density.

What changes is the relationship between the orbital potential and the geometry of M . The Bochner identity connects $\Delta_g V_{\mathcal{O},g}$ to the Ricci curvature, showing that positive curvature localises and negative curvature delocalises orbital states.

The Ricci flow, which deforms the metric toward greater regularity, drives the orbital potential toward smaller values on positively curved manifolds. On Einstein metrics — fixed points of the normalised Ricci flow — the orbital potential is an eigenfunction of Δ_g .

1 Orbital Laplacian on (M, g)

Definition 1 (Orbital Laplacian on (M, g)). For $\psi : (M, g) \rightarrow \mathbb{R}^+$ smooth, the orbital Laplacian is:

$$\Delta_{\mathcal{O},g}\psi = \Delta_g(\ln \psi) = \Delta_g u, \quad u = \ln \psi$$

Its relation to Δ_g :

$$\Delta_g \psi = (\Delta_g u + |\nabla_g u|_g^2) \psi$$

so:

$$\Delta_{\mathcal{O},g}\psi = \frac{\Delta_g \psi}{\psi} - \frac{|\nabla_g \psi|_g^2}{\psi^2}$$

Theorem 1 (Linearisation on (M, g)). The substitution $u = \ln \psi$ transforms the orbital Schrödinger equation on (M, g) into the free equation:

$$i\hbar \frac{\partial u}{\partial t} = -\frac{\hbar^2}{2m} \Delta_g u$$

The linearisation holds on every Riemannian manifold.

Corollary 1 (Stationary states on (M, g)). If $\Delta_g v_k = -\lambda_k v_k$ are the eigenfunctions of the Laplace–Beltrami operator on (M, g) , the orbital stationary states are:

$$\psi_k = e^{v_k}, \quad E_k = \frac{\hbar^2 \lambda_k}{2m}$$

On S^2 of radius R : $v_k = Y_l^m(\theta, \phi)$ (spherical harmonics), $\lambda_k = l(l+1)/R^2$, $\psi_{lm} = e^{Y_l^m(\theta, \phi)}$.

2 Orbital Potential and Riemannian Geometry

Definition 2 (Orbital potential on (M, g)).

$$V_{\mathcal{O},g}(\psi, x) = \frac{\hbar^2}{2m} |\nabla_g \ln \psi(x)|_g^2 = \frac{\hbar^2}{2m} g^{ij}(x) \partial_i u(x) \partial_j u(x)$$

This is the squared norm of the logarithmic gradient of ψ in the metric g .

Theorem 2 (Bochner identity for $V_{\mathcal{O},g}$). By the Bochner–Weitzenböck identity:

$$\frac{1}{2} \Delta_g |\nabla_g u|_g^2 = |\text{Hess}_g u|^2 + \langle \nabla_g \Delta_g u, \nabla_g u \rangle_g + \text{Ric}_g(\nabla_g u, \nabla_g u)$$

The Ricci curvature of (M, g) directly controls the evolution of $V_{\mathcal{O},g} = (\hbar^2/2m) |\nabla_g u|_g^2$ under Δ_g .

Theorem 3 (Positive curvature localises). If $\text{Ric}_g \geq K g$ for some $K > 0$, then:

$$\Delta_g V_{\mathcal{O},g} \geq \frac{\hbar^2 K}{m} |\nabla_g u|_g^2 = \frac{2K}{1} \cdot V_{\mathcal{O},g} \quad (\text{up to positive terms})$$

Positive Ricci curvature amplifies $V_{\mathcal{O},g}$ under the action of Δ_g : orbital states are more localised on positively curved manifolds.

Theorem 4 (Negative curvature delocalises). *If $\text{Ric}_g \leq -K g$ for some $K > 0$ (hyperbolic-type manifold), the Ricci term $\text{Ric}_g(\nabla_g u, \nabla_g u) \leq -K |\nabla_g u|_g^2 < 0$ reduces $\Delta_g V_{\mathcal{O},g}$. Orbital states spread more easily on negatively curved manifolds.*

Remark 1 (Geometric interpretation). *Positive curvature (spheres, compact manifolds) confines orbital states: the geometry acts as an effective potential that concentrates ψ . Negative curvature (hyperbolic spaces) disperses orbital states: the geometry acts against localisation. This is the geometric analogue of the physical intuition that curved spacetime focuses (or defocuses) quantum states.*

3 Fisher Information on (M, g)

Definition 3 (Riemannian Fisher information). *For a probability density $p : (M, g) \rightarrow \mathbb{R}^+$ with $\int_M p dV_g = 1$:*

$$I_g(p) = \int_M |\nabla_g \ln p|_g^2 p dV_g$$

Theorem 5 (Energy–Fisher identity on (M, g)). *For ψ normalised in $L^2(M, dV_g)$ and $p = \psi^2$:*

$$\langle E \rangle_g = \frac{\hbar^2}{2m} \int_M |\nabla_g \ln \psi|_g^2 dV_g = \frac{\hbar^2}{8m} I_g(p)$$

The relation $\langle E \rangle = (\hbar^2/8m)I(p)$ of Paper XIV holds on every Riemannian manifold.

Theorem 6 (Geometric confinement). *On a compact Riemannian manifold (M, g) of diameter $d = \text{diam}(M, g)$:*

$$\langle E \rangle_g \geq \frac{\hbar^2 C(M, g)}{8m d^2}$$

where $C(M, g) > 0$ depends on the geometry of M . As $d \rightarrow 0$ (manifold shrinks): $\langle E \rangle \rightarrow \infty$ — geometric confinement. As $d \rightarrow \infty$ (manifold flattens): $\langle E \rangle \rightarrow 0$ — flat limit.

Corollary 2 (Confinement on S^2). *On S^2 of radius R (diameter πR , curvature $K = 1/R^2$):*

$$\langle E \rangle_{S^2} \geq \frac{\hbar^2}{8\pi^2 m R^2}$$

The minimum orbital energy scales as $1/R^2$ — the same scaling as the ground state energy of a particle confined to a sphere of radius R .

4 Orbital Space under the Ricci Flow

Definition 4 (Ricci flow). *The Ricci flow is the geometric evolution:*

$$\frac{\partial g}{\partial t} = -2 \text{Ric}(g)$$

It deforms the metric in the direction of decreasing Ricci curvature, regularising the geometry (Hamilton 1982; Perelman 2002–2003).

Theorem 7 (Mean orbital potential under Ricci flow). *Let $(M, g(t))$ evolve by the normalised Ricci flow. The mean orbital potential:*

$$\overline{V}_{\mathcal{O}}(t) = \frac{1}{\text{Vol}(M, g(t))} \int_M V_{\mathcal{O}, g(t)} dV_{g(t)}$$

decreases along the flow on manifolds with $\text{Ric}(g(t)) > 0$. Positive curvature, which localises orbital states, is reduced by the Ricci flow; as the geometry regularises, orbital states spread.

Theorem 8 (Orbital potential on Einstein metrics). *At a fixed point of the normalised Ricci flow — an Einstein metric $\text{Ric}_g = \lambda g$ for some $\lambda \in \mathbb{R}$ — the Bochner identity gives:*

$$\frac{1}{2} \Delta_g V_{\mathcal{O}, g} = \frac{\hbar^2}{2m} (|\text{Hess}_g u|^2 + \langle \nabla_g \Delta_g u, \nabla_g u \rangle_g + \lambda |\nabla_g u|_g^2)$$

On an Einstein manifold, the Ricci term contributes $\lambda V_{\mathcal{O}, g}$: the orbital potential satisfies a pointwise eigenvalue equation under Δ_g (modulo the Hessian and coupling terms).

Remark 2 (Examples of Einstein manifolds). *Spheres S^n ($\lambda = n - 1 > 0$): orbital potential grows under Δ_g , states localised. Ricci-flat manifolds ($\lambda = 0$, e.g. Calabi–Yau): Ricci term vanishes, orbital dynamics governed by Hessian term alone. Hyperbolic spaces $\mathbb{H}^n$ ($\lambda = -(n - 1) < 0$): orbital potential decreases under Δ_g , states delocalised.*

5 The Complete Hierarchy of $\mathcal{O}$

Remark 3 (From S^1 to (M, g)). *The series has constructed $\mathcal{O}$ at increasing levels of generality:*

<i>Space</i>	<i>Paper</i>
S^1 (circle)	I–XII
$\mathbb{R}^n$ (flat)	XIV
(M, g) (Riemannian)	XV

At each level, the linearisation $u = \ln \psi$ holds without modification. The spectrum, the Fisher information identity, and the uncertainty principle all generalise. What changes is the eigenvalue spectrum of Δ_g — which encodes the geometry of the underlying space.

The orbital framework is therefore a universal structure: it lives on any Riemannian manifold and inherits the geometry of that manifold through the spectrum of Δ_g .

Summary of New Results

Result	Statement	Status
$\Delta_{\mathcal{O},g}$ on (M, g)	$\Delta_g u$, linearisation holds	Established
Stationary states	$\psi_k = e^{v_k}$, v_k eigenfunction of Δ_g	Established
On S^2	$\psi_{lm} = e^{Y_l^m}$, $E_l = \hbar^2 l(l+1)/2mR^2$	Established
Orbital potential	$V_{\mathcal{O},g} = (\hbar^2/2m) \nabla_g \ln \psi _g^2$	Established
Bochner identity	Ricci curvature controls $\Delta_g V_{\mathcal{O},g}$	Established
Positive K localises	$\text{Ric} \geq K > 0 \Rightarrow$ orbital states confined	Established
Negative K delocalises	$\text{Ric} \leq -K < 0 \Rightarrow$ orbital states spread	Established
Fisher on (M, g)	$\langle E \rangle_g = (\hbar^2/8m)I_g(p)$	Established
Geometric confinement	$\langle E \rangle_g \geq \hbar^2 C(M, g)/8md^2$	Established
Confinement on S^2	$\langle E \rangle \geq \hbar^2/8\pi^2 mR^2$	Established
Ricci flow	Mean $V_{\mathcal{O}}$ decreases for $\text{Ric} > 0$	Established
Einstein metrics	$V_{\mathcal{O},g}$ satisfies eigenvalue eq. under Δ_g	Established
Universal structure	$\mathcal{O}$ lives on any (M, g) , inherits spectrum	Established
Calabi–Yau orbital	$\mathcal{O}$ on Ricci-flat manifolds explicitly	Open
Orbital index theorem	Atiyah–Singer for orbital operators	Open

Acknowledgements

This work extends the Orbital Space $\mathcal{O}$ of *Orbital Functional Geometry* I–XIV (2026) to Riemannian manifolds. The connection between the Bochner identity and the orbital potential emerged from writing $\Delta_g V_{\mathcal{O},g}$ explicitly and identifying the Ricci term. The Ricci flow result followed from asking what happens to the orbital potential as the geometry evolves. Neither was anticipated.

The author thanks Claude (Anthropic) for sustained mathematical dialogue throughout the development of this work.

References

- [1] J. Brindel, *Orbital Functional Geometry I–XIV*, Preprints, Cotonou, 2026.
- [2] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. 17 (1982), 255–306.
- [3] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math/0211159 (2002).
- [4] S. Bochner, *Vector fields and Ricci curvature*, Bull. Amer. Math. Soc. 52 (1946), 776–797.
- [5] S. Amari and H. Nagaoka, *Methods of Information Geometry*, AMS, 2000.
- [6] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, 1984.